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Study of the Features and Properties of Greens Functions of Macroscopic Quantum Systems

This article is devoted to the theoretical investigation of processes in macroscopic systems based on wave-particle duality and quantum statistical distributions using the Green's function method. In particular, it provides a detailed description of the properties and characteristics of the Green's function for an ideal Fermi liquid, single- and two-particle Green's functions, as well as Green's functions at limiting temperatures and for the Dirac and Schrödinger equations. The study analyzes the application of the Green's function method to processes in many-particle quantum systems and to solving equations that describe the constituent particles of such systems. For a given operator and set of boundary conditions, a Green's function can be defined, allowing the solution of the corresponding equation to be expressed in integral form. A distinctive feature of this method is the application of mathematical-physics tools, such as Fourier methods, integral transforms, special functions, and mathematical analysis, to problems in quantum physics governed by the principles of quantum and statistical mechanics. The relevance of this study stems from the application of the Green's function method to the analysis of quantum phenomena. This approach is considered one of the effective methods for solving partial differential equations describing various physical processes. It involves transforming the original differential equation into an equivalent integral equation involving Green's functions, which can then be solved. The transformation involves passing to the momentum representation and applying Fourier transforms using methods of complex analysis.

Keywords: Fermi liquid, Fourier transform, momentum representation, macroscopic system, Green's function, integral equation

Introduction

Quantum liquids, first discovered in 1938 by P. Kapitsa and J.F. Allen, represent macroscopic systems governed by quantum effects (such as energy quantization, wave-particle duality, and quantum statistical distributions). In quantum mechanics, it is possible to describe the states of individual particles and their interactions; however, in quantum liquids containing a large number of particles, these particles cannot be considered independently. Their quantum properties manifest only through collective processes. Superfluid helium-4 and Fermi liquids formed by fermions at low temperatures are among the primary objects of study in hydrodynamics, and their nature fully satisfies the criteria of quantum liquids. In such media, macroscopic quantum phenomena predominantly occur. Electrons in metals and semiconductors, excitons in dielectrics, and nucleons in atomic nuclei can also be treated as quantum liquids.

In 1830, the British mathematician George Green, while solving inhomogeneous boundary value problems in electrostatics and heat conduction, introduced the functions that now bear his name and established the foundations of their theoretical framework. These functions, which serve as a fundamental mathematical tool in mathematical physics, were further developed throughout the 20th century and significantly simplified the solution of linear inhomogeneous differential equations with boundary conditions in electrodynamics, quantum field theory, and scattering theory. In particular, in quantum field theory, Green's functions describe the probability amplitude for a transition from one point in space-time to another, whereas in scattering theory they characterize particle scattering by various objects. In the theory of fluctuations, Green's functions also facilitate the construction of Feynman diagrams for different quantum processes.

At the same time, Green's functions in quantum liquids serve as a powerful mathematical framework for describing many-body systems. They make it possible to evaluate the influence of external perturbations on a system, analyse its response to such perturbations, and determine its thermodynamic and dynamic properties, thereby providing information about both equilibrium and non equilibrium states. In addition, Green's functions are widely used to calculate physical quantities such as density, correlation functions, and quasiparticle spectra. Problems related to finite temperature in Fermi liquids are addressed using temperature-dependent Green's functions that incorporate thermal effects. Meanwhile, processes involving the motion of a single particle among many others are described using single-particle Green's functions, whereas two-particle Green's functions characterize correlations between particles. Considering liquid helium at low temperatures as a quantum liquid makes it possible to establish theoretical principles and conditions for applying Green's function methods to the study of thermodynamic and dynamic properties of interacting quantum systems. This approach enables the calculation of single-particle and many-particle correlation functions, the determination of thermodynamic spectra, and the analysis of the relationship between microscopic interactions and macroscopic properties. The main subject of investigation is quantum liquids whose properties are governed by quantum effects. In this context, the study addresses the following key problems: many-body quantum systems, the concept of quantum liquids, neutral Fermi liquids and their equilibrium properties, as well as kinetic equations for quasiparticles in Fermi liquids.

When studying quantum liquids using Green's functions, it becomes possible to account for particle interactions within the liquid. This approach allows one to determine the Green's function corresponding to a specific operator and boundary conditions. Using this function, the solution of the equation describing the liquid can be expressed in an integral form. In solving such problems of quantum macrophysics, which are based on the principles of quantum mechanics and statistical mechanics, various mathematical methods are employed, including Fourier methods, integral transform techniques, special functions, and methods of mathematical analysis. In addition, the following mathematical tools of mathematical physics are widely used: partial differential equations, integral equations, functional analysis, and probability theory. Variational methods also play an important role in the study, as they are used to determine stationary states of physical systems.

A brief review of recent scientific studies relevant to the topic of this paper can be summarized as follows. In [1–3], a comprehensive theoretical study of radio-frequency spectra of a two-component Fermi gas with balanced populations in the normal phase of the temperature–interaction phase diagram was carried out. Analytical results were also obtained for determining the shape and width of radio-frequency spectra within relevant temperature and interaction regimes. In [4], the problem of determining the Green's function for the stationary Schrödinger equation was considered within the framework of the potential quantization method. An expression for Green's functions in terms of eigenfunctions and eigenvalues was derived, and the equivalence between this approach and the traditional method was demonstrated. Studies [3, 4] present an independent review of the theory of quantum melting mediated by dislocations at zero temperature in two spatial dimensions. A one-dimensional gas of spinful fermions interacting via an attractive δ -function potential was also examined using the Bethe ansatz solution. In [5–7], the application of the Bogoliubov–Tyablikov two-time Green's function method to the many-electron theory of metals is discussed within strongly correlated models, such as the Hubbard model and the Vonsovskiy exchange model. Issues such as atomic representations of electronic states, the metal–insulator transition, incoherent states, and the Kondo effect are analysed. The criterion for ferromagnetism in systems of strongly correlated electrons has been analysed, and it is shown to differ fundamentally from the conventional condition. Based on the analysis of radiative recombination spectra of two-dimensional electrons with photoexcited holes bound to remote acceptors, the dependence of the quasiparticle Fermi energy on electron density has been investigated. The dependence of the renormalized quasiparticle mass on the concentration of two-dimensional electrons has been measured. It has been shown that, in a perpendicular magnetic field, the quasiparticle concept in a two-dimensional Fermi liquid remains valid not only near the Fermi surface but also deep below it, down to the bottom of the size-quantization subband, since the excitation broadening is significantly smaller than their energy [8]. In, a one-dimensional gas of spin-1/2 fermions interacting via a repulsive δ -function potential of arbitrary strength is considered. For the case where all fermions except one have spin up, the time-dependent two-point correlation function of the spin-down fermion is calculated. This impurity Green's function is represented in the thermodynamic limit as an integral over Fredholm determinants of integrable linear integral operators. The problem of integer and fractional quantum Hall resistance within a unified framework of mathematical physics has remained an unresolved issue over the past four decades. In this approach, the quantum Hall liquid is

modelled as a membrane of electron fluid flowing around magnetic fluxes. The trajectories of fractional quantum Hall states lie along the edge surface of the cross-section of a folded membrane, which is spontaneously characterized by a lattice of magnetic fluxes in the bulk zone. This topological representation provides an alternative way for the realization of string theory concepts and topological quantum computation through the folding of a laminar quantum liquid. For a system to be considered a quantum liquid, quantum statistical effects must be present in addition to quantum mechanical effects [9]. An unexplored dynamical condition for the manifestation of quantum statistical effects is considered, where particle exchange events in a gas-like hydrodynamic regime are treated as a dynamical process characterized by an intrinsic time scale. Subsequently, quantitative criteria for particle indistinguishability and the associated quantum statistics are proposed, which are effective over short time scales and manifest over longer times. A quantum critical point is identified, and hypotheses regarding the formation of complex quasiparticle condensates are proposed. In [10–12], phenomena arising in fermions in optical lattices at low temperatures under an external magnetic field are investigated. By tuning the field, the attractive interaction between any two fermions can be made arbitrarily strong, leading to the formation of composite bosons via so-called Feshbach resonances. At sufficiently low temperatures, a critical temperature is determined at which these composite bosons undergo Bose–Einstein condensation, resulting in the formation of massive fermions. A quantum critical point is identified, and hypotheses regarding the formation of complex quasiparticle condensates are proposed. Quantum-critical behaviour induced by a presumed quantum phase transition separating the d-wave superconducting state is investigated in detail. Within the framework of the renormalization group approach, a phenomenological effective theory based on the Landau–Ginzburg–Wilson theory is employed, and one-loop calculations are performed to construct a consistent set of coupled interaction parameters [11]. A coupled fermion–boson system in spatial dimensions is investigated, and a new fixed point for a non-Fermi liquid is identified [13].

The peculiarity of describing macro systems using Green’s functions is that they reduce complex multi-particle problems to the search for single-particle responses. This allows for the accurate consideration of particle statistics (Bose or Fermi) and their interactions, representing the physical properties of the system through integral equations. The Green’s function acts as a “propagator” that describes the probability of a particle added to the system at one point being detected at another point after some time. If Green’s functions in quantum field theory describe the probability of passage between points in space-time, then in scattering theory they were able to analytically explain the scattering of particles from various objects. At the same time, in the theory of perturbations, Green’s functions also helped to construct Feynman diagrams for various quantum processes, and Green’s functions for quantum fluids were used as a mathematical tool for describing the properties of many-body systems. In short, with the help of these functions, it was possible to systematically evaluate the influence of external influences on the system, the behavior of the system in relation to this influence, and to analytically consider the thermodynamic and dynamic properties of the system. It was possible to obtain accurate information from the conditions of the system in equilibrium and non-equilibrium states. Today, Green’s functions are successfully used in calculating important quantities such as density, correlation functions, quasiparticle spectra, etc. While problems related to the temperature of a Fermi liquid are solved using temperature Green’s functions that take into account thermal effects, and problems related to the motion of an individual particle between other particles are solved using single-particle Green’s functions, two-particle Green’s functions can determine the correlation between two particles. All key thermodynamic and kinetic properties are expressed through the Green’s function: density of states, binding energies, conductivity, magnetic susceptibility, and excitation spectra. The interaction of particles is taken into account by summing up infinite perturbation theory series using Feynman diagrams. The Green’s function acts as a “propagator”—the probability that a particle will move from one point to another, interacting with its surroundings.

In the present work, the features and properties of Green’s functions for an ideal Fermi liquid are examined in detail, including single-particle and two-particle Green’s functions, as well as Green’s functions at limiting temperatures and for the Dirac and Schrödinger equations. The application of Green’s function methods to processes occurring in many-body quantum systems and to the solution of equations describing particles forming such systems is analysed. For a given operator and boundary conditions, the corresponding Green’s function is determined, and the solution of the equation describing the liquid is expressed in integral form using this function. A distinctive feature of this approach is that, in solving such problems of quantum macrophysics based on the principles of quantum mechanics and statistical mechanics, a combination of mathematical methods is employed, including Fourier methods, integral transform techniques, special func-

tions, and methods of mathematical analysis. The relevance of this study is associated with the application of Green's function methods to quantum phenomena. This approach is considered one of the most effective tools for solving partial differential equations describing various physical processes. In particular, it allows the transformation of a differential equation describing a given phenomenon into an equivalent integral equation incorporating Green's functions, which can then be solved.

The main problem is the calculation of the ideal Fermi-liquid, One-and two-particle Green functions, as well as the limit temperature Green functions, the Green functions of the Dirac and Schrodinger equations. Determining solutions to equations for multi-particle quantum systems by the Green function approach is also one of the most pressing problems. Ways of transition from a differential equation to an integral equation equivalent to this equation were considered and theoretical searches were made in the direction of determining the solution of this equation according to the Green function.

Experimental

Quantum macrophysics can be regarded as a distinct branch of theoretical physics that studies phenomena in solid and liquid phases based on the principles of quantum mechanics, statistical mechanics, and electrodynamics within a many-particle framework. The nature of these phenomena is investigated using appropriate mathematical models. For example, a deeper understanding of the properties of metals requires treating electrons as a Fermi liquid. Since the presence of a crystal lattice complicates the development of a theory of electron liquids, it becomes necessary to establish general principles for homogeneous and isotropic systems. Similarly, superconductivity in metals can be explained using a simplified isotropic model of a superfluid electron liquid. In the study of macromolecules (many-body systems, macroscopic quantum systems), the Green's function method is used to describe the system's response to external influences, calculate thermodynamic properties, and search for excitation spectra (quasiparticles). The method of Green's functions, which depend on imaginary time or temperature, is adapted to describe equilibrium statistical mechanics.

At sufficiently low temperatures (1-2 K), where the de Broglie wavelength associated with the thermal motion of atoms becomes comparable to interatomic distances, macroscopic phenomena in liquids are governed by quantum laws. According to classical theory, all substances should exist in a solid state at absolute zero. However, due to weak interatomic interactions, the liquid isotopes of helium, ^3He and ^4He remain in the liquid state near absolute zero temperature, where quantum phenomena are observed. In such cases, the solid phase becomes energetically less favourable. The method of equations of motion uses the left-hand chain of equations for the Green's functions, where special splitting factors are used to close the system. The concept of Fermi liquids considers systems with strong interactions, such as electrons in metals. The Green's function is used here to justify the concept of quasiparticles.

Results and Discussion

If we consider a macroscopic system composed of particles with half-integer spin within the Heisenberg representation—where wave functions are time-independent and operators f , corresponding to physical observables depend explicitly on time and can be written as follows:

$$f_a(t, \vec{r}) = f_a(\vec{r}) \exp(iHt) \exp(-iHt);$$

$$f_a^*(t, \vec{r}) = f_a^*(\vec{r}) \exp(iHt) \exp(-iHt).$$

Then, the single-particle Green's function of the system can be written as:

$$G_{ab}(Z_1, Z_2) = -i \overline{N f_a(Z_1) f_b^*(Z_2)}.$$

Here, the variables $t, \vec{r}$ are replaced by $t_1, \vec{r}$ and $t_2, \vec{r}$, respectively, and are denoted collectively as Z_1, Z_2 ; N is the time-ordering operator; and the overline indicates averaging over the ground state of the system. In quantum mechanics and quantum field theory, the Green's function is the impulse response of a quantum-mechanical operator (e.g., the Hamiltonian) that describes the propagation of a particle and its response to an external stimulus. When a particle has spin, its wave function is a spinor (magnetic vector), and the Green's function acquires a matrix-spinor structure. The spin dependence of the Green's function in the absence of an external field is described by a second-rank spinor δ_{ab} :

$$G_{ab}(Z_1, Z_2) = \delta_{ab} G(Z_1, Z_2).$$

The coordinate-space density matrix of particles in the system is given by [1]:

$$\rho_{ab}(\vec{r}_1, \vec{r}_2) = n^{-1} \overline{f_b^*(t, \vec{r}_2)} f_a(t, \vec{r}_1). \quad (1)$$

Using this expression, one can determine the expectation value of any physical observable for an individual particle. Let us rewrite Eq. (1) in terms of the Green's function:

$$\rho_{ab}(\vec{r}_1, \vec{r}_2) = i n^{-1} G_{ab}(t_1, \vec{r}_1; t_1 \rightarrow 0, \vec{r}_2).$$

The Fourier expansion of this function allows one to determine the momentum distribution of particles:

$$n(\vec{p}) = n \int \rho(\vec{r}_1, \vec{r}_2) \exp\{-i\vec{p}(\vec{r}_1 - \vec{r}_2)\} d^3(x_2 - x_1) = -i \int G(t, \vec{r}) \exp\{-i\vec{p}\vec{r}\} d^3x. \quad (2)$$

This expression represents the concentration of real (i.e., non-quasiparticle) particles within a given momentum range $\frac{d^3p}{(2\pi)^3}$. Equation (2) should not be confused with the quasiparticle distribution function. If the expression for the Green's function in the momentum representation is substituted into Eq. (2), i.e.

$$G(t, \vec{r}) = \int G(\omega, \vec{p}) \exp\{i(\vec{p}\vec{r} - \omega t)\} \frac{d\omega d^3p}{(2\pi)^4},$$

then Eq. (2) takes the following form:

$$n(\vec{p}) = -i \lim_{t \rightarrow 0} \int_{-\infty}^{\infty} G(\omega, \vec{p}) \exp\{-i\omega t\} \frac{d\omega}{2\pi},$$

where ω is a complex variable,

$$G(\omega, \vec{p}) = \int G(t, \vec{r}) \exp\{-i(\vec{p}\vec{r} - \omega t)\} dt d^3x. \quad (3)$$

Taking into account the discontinuity property of the Green's function $G(t, \vec{r})$, and the integration over dt in Eq. (3), the integral can be split into two parts with limits from $-\infty$ to and from 0 to ∞ . Then, using the rules of matrix multiplication, we obtain the following expression m for particle states of type a :

$$G(t, \vec{r}) = -\frac{i}{2} \sum_m |f_a(Z_1)| \overline{m \cdot m} |f_a^*(Z_2)|.$$

It is known that Heisenberg operators are physical observables whose mathematical description changes over time, while the quantum state of the system itself remains unchanged. Therefore, we can say that these operators are responsible for the time-varying states of a free particle in the system. The summation is performed over all quantum states of the system. The Heisenberg operator f for a system of free particles satisfies the equation:

$$i \frac{\partial f_a}{\partial t} = -\frac{1}{2m} \Delta f_a - \mu f_a.$$

Taking into account the above results, this equation can be written in terms of the Green's function as:

$$\left(i \frac{\partial}{\partial t} = -\frac{\Delta}{2m} + \mu \right) G(t, \vec{r}) = \delta(t) \delta(\vec{r}),$$

where μ is the chemical potential of the liquid [2]. The Fourier transform is a mathematical bridge between a particle's wave function in coordinate representation and its momentum representation. It allows equations to be translated from the spatial domain to the frequency domain. After performing the Fourier transform, we obtain:

$$\left(\omega - \frac{\vec{p}^2}{2m} + \mu \right) G_0(\omega, \vec{p}) = 1.$$

G_0 is the Green's function for non-interacting particles.

$$G_0(\omega, \vec{p}) = \left[\omega - \frac{\vec{p}^2}{2m_0} + \mu + i \text{sign} \omega \right]^{-1}.$$

This function is a mathematical tool (probability amplitude) that describes the propagation of a quantum particle in space and time.

In quantum mechanics, the product of operators means their sequential application to the wave function (the state of the system). It is used to mathematically describe multi-step measurements, complex physical

quantities, and how a system changes over time. The following product of four Heisenberg operators f averaged over the ground state N is called the two-particle Green's function:

$$Q_{34,12} = \overline{N f_3 f_4 f_1^* f_2^*}. \quad (4)$$

It is convenient to consider this function in the momentum representation for a homogeneous system, i.e., it depends only on the differences of three arguments

$$Z_3 - Z_2, Z_4 - Z_2, Z_1 - Z_2.$$

This is due to the fact that the Fourier transform components, taken over all variables Z_1, Z_2, Z_3, Z_4 , contain a δ -function:

$$\begin{aligned} \int Q_{34,12} \exp\{i(p_3 Z_3 + p_4 Z_4 - p_1 Z_1 - p_2 Z_2)\} d^4 Z_1 d^4 Z_2 d^4 Z_3 d^4 Z_4 = \\ = (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) Q_{\delta,ab}(p_3, p_4, p_1, p_2). \end{aligned}$$

For the arguments of the Green's function, the following relation holds $p_1 + p_2 = p_3 + p_4$. In the zeroth-order approximation, this function can be expressed in terms of differences of single-particle Green's functions [2–5]:

$$Q_{\delta,ab}^0(p_3, p_4, p_1, p_2) = (2\pi)^4 \left[\delta^{(4)}(p_1 - p_3) G_a^0(p_1) G_b^0(p_2) - \delta^{(4)}(p_1 - p_4) G_b^0(p_2) G_a^0(p_1) \right].$$

In higher-order approximations, additional terms appear that are not included in the Green's function.

$$\begin{aligned} Q_{A_3 A_4, A_1 A_2}^0 = (2\pi)^4 \left[\delta^{(4)}(p_1 - p_3) G_{A_3 A_1}^0(p_1) G_{A_4 A_2}^0(p_2) - \delta^{(4)}(p_1 - p_4) G_{A_3 A_2}^0(p_2) G_{A_4 A_1}^0(p_1) \right] + \\ + G_{A_3 B_3}(p_3) G_{A_4 B_4}(p_4) i R_{B_3 B_4, B_1 B_2}(p_3, p_4, p_1, p_2) G_{B_1 A_1}(p_1) G_{B_2 A_2}(p_2). \end{aligned}$$

Here, the particle components ab of the system are replaced by AB . The function R in the last expression is called the vertex function and, in the momentum representation, can be written as follows:

$$R_{\delta a,ab}(p_3, p_4, p_1, p_2) = -R_{\delta b,ab}(p_4, p_3, p_1, p_2) = -R_{\delta b,ba}(p_3, p_4, p_2, p_1).$$

To determine the Green's function at nonzero temperature for a macroscopic system, it is necessary to replace averaging over the ground state of a closed system with averaging over ($\langle \rangle$) the Gibbs distribution [6–10]. When analysing the properties of Green's functions at limiting temperatures, it is important to introduce the *retarded* and *advanced* Green's functions. The retarded Green's function is defined as follows:

$$iG_{ab}^\alpha(Z_1, Z_2) = \langle f_a(Z_1) f_b^*(Z_2) + f_b^*(Z_2) f_a(Z_1) \rangle.$$

In the absence of an external field, this function reduces to a scalar function that depends only on the difference of its arguments: $Z = Z_1 - Z_2$:

$$G_{ab}^\alpha(Z_1, Z_2) = \delta_{ab} \frac{1}{2} G_{aa}^\alpha.$$

Since $G^\alpha(t, \vec{r}) = 0$ for $t < 0$, the momentum representation of the above function can be written as follows:

$$G^\alpha(\omega, \vec{p}) = \int_0^\infty \int \exp i(\omega t - \vec{p} \vec{r}) G^\alpha(t, \vec{r}) dt d^3 z.$$

Similarly, the advanced Green's function can be obtained in the same manner G^β . The temperature Green's function is a mathematical apparatus of quantum statistics that describes the behavior of a quantum many-body system when taking into account temperatures different from absolute zero. Then, in our case, the temperature-dependent Green's function is given by:

$$G(\omega, \vec{p}) = -\frac{(2\pi)^3}{2} \sum_{n,m} \omega_n A_{mn} \delta(\vec{p} - \vec{k}_{nm}) \left\{ \frac{1}{\omega_{nm} - \omega} \left(1 + e^{-\frac{\omega_{nm}}{T}} \right) \right\}.$$

Application of Green's Functions to the Solution of the Dirac Equation. For the calculation of processes within the Standard Model, it is necessary to employ methods of mathematical physics and the theory of special functions, since the equations describing scalar, vector, and spinor particles are differential equations of different orders. Solving the Dirac equation using the Green's function allows us to reduce a complex differential problem (especially in the presence of external fields or potentials) to an integral equation. This is a standard technique in relativistic quantum mechanics and quantum field theory, where Green's functions are

also called propagators. As an illustrative example, we consider the solution of the Dirac equation given below for an electron in an electromagnetic field using the Green's function method [11–13]:

$$(i\sigma^\alpha \partial_\alpha - m)\varphi(x) = -e\sigma^\alpha E_\alpha(x)\varphi(x), \quad (5)$$

where $\sigma^\alpha = \{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$ are the Dirac matrices, and e is the electron charge. To solve Eq. (5), it is necessary to first consider the following auxiliary problem:

$$(i\sigma^\alpha \partial_\alpha - m)G(x, x^0) = \delta^{(4)}(x - x^0).$$

Then, the solution of Eq. (5) can be represented as:

$$\varphi(x) = -e \int G(x, x^0) \sigma^\alpha E_\alpha(x^0) \varphi(x^0) dx^0.$$

The momentum representation of Green's functions (or the momentum space representation) is a transition from space-time coordinates to wave vectors (momenta) and frequencies using the Fourier transform. The momentum representation of the Green's function in this expression is given by:

$$G(x, x^0) = \frac{1}{(2\pi)^4} \int S(p) \exp\{-ip(x - x^0)\} d^4 p. \quad (6)$$

The momentum representation of Green's functions is a mathematical apparatus in which the solution of quantum mechanical equations (Schrödinger or Dirac) is expressed in terms of particle momenta rather than their coordinates. This significantly simplifies the analysis due to the translational invariance of space. $S(p)$ in expression (6) is the Fourier transform of the Green's function. In the Fourier representation, the Green's function is transformed from a complex integral convolution kernel into a simple algebraic function. A differential equation with a source is reduced to dividing the Fourier transform of the source by the Fourier transform of the differential operator. If the Dirac delta function is applied,

$$\delta^{(4)}(x - x^0) = \frac{1}{(2\pi)^4} \int \exp\{-ip(x - x^0)\} d^4 p$$

and the notation $\sigma^\alpha p_\alpha = p$, is introduced, as well as the commutation property of the Dirac matrices $\sigma^\alpha \sigma^\beta + \sigma^\beta \sigma^\alpha = 2\theta^{\alpha\beta}$, then the expression for the Green's function [14], corresponding to the Fourier-transformed Green's function and the Dirac equation, takes the following form:

$$S(p) = \frac{\tilde{p} + m}{p^2 - m^2};$$

$$G(x, x^0) = \frac{1}{(2\pi)^4} \int \frac{\tilde{p} + m}{p^2 - m^2 + i\varepsilon} \exp\{-ip(x - x^0)\} d^4 p.$$

It is well known that a particle in an external field in quantum systems is described by the Schrödinger equation:

$$(\hat{H} - E)\psi(\vec{r}) = -\hat{F}\psi(\vec{r}), \quad (7)$$

where $\hat{F}$ is the operator responsible for the interaction, and $\hat{H}$ is the Hamiltonian,

$$\psi(\vec{r}) = \exp(i\vec{k}\vec{r}), \quad (8)$$

to solve Eq. (7) by the Green's function method, it is necessary to pass from this differential equation to an equivalent integral equation (that is, an integral term entering the Green's function is added to the wave function of the form given in Eq. (8)).

$$\psi(\vec{r}) = \exp(i\vec{k}\vec{r}) + \int G(E, \vec{r}, \vec{r}') \hat{F}(\vec{r}) \psi(\vec{r}') d\vec{r}'.$$

Here, $G(E, \vec{r}, \vec{r}')$ is the Green's function corresponding to the Hamiltonian operator and satisfying the equation given below $(E - \hat{H})G(E, \vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}')$ with the Dirac delta function ($\delta(\vec{r} - \vec{r}')$) [15]. Its spectral form is given by:

$$G(E, \vec{r}, \vec{r}') = \sum_{n=1}^{\infty} \frac{\psi_n(\vec{r}) \psi_n^*(\vec{r}')}{E_0 - E_n}.$$

In the case of a continuous spectrum,

$$G(E, \vec{r}, \vec{r}') = \int \frac{\psi_n(\vec{r}) \psi_n^*(\vec{r}')}{E_0 - E_n} \frac{d\vec{k}}{(2\pi)^3}.$$

It should be noted that the form of the Green's functions entering the equivalent integral equations, in momentum and spectral representations as well as in Fourier-transformed form, makes it possible to accurately determine the domains of definition of the processes under consideration in phase space. On the other hand, this approach improves the assessment of the rate of the process in combination with the theory of functions of a complex variable. The obtained results can be used in solving problems based on scattering theory and in calculating quantum-field amplitudes of spinor particles.

Green's Functions of Averages and Equilibrium Green's Functions. When expressing the expectation values of quantities corresponding to quantum operators in terms of known Green's functions, only the expectation values of two field operators and single-particle operators can appear in such expressions [16, 17]. For example, the particle number density in a system is given by:

$$n(\vec{r}) = \sum_a \langle \psi_a^*(\vec{r}) \psi_a(\vec{r}) \rangle = \pm \sum_a G_{aa}(\vec{r}, t - 0; \vec{r}, t)$$

and for a homogeneous system of fermions:

$$n = \frac{N}{V} = -2iG_{aa}(\vec{r} = 0, t = -0). \quad (9)$$

This relation, which connects the Green's function with the particle density, makes it possible to express the thermodynamic parameters of the system under $T = 0$ in terms of Green's functions. Indeed, the thermodynamic potential of the system under the conditions $T = 0$, $V = \text{const}$ and $S = 0$ is determined by the relation:

$$dF = -SdT - pdV - Nd\mu = -Nd\mu.$$

This relation can be integrated subject to the condition $F(\mu = 0) = 0$:

$$F = \int_0^\mu d\mu N(\mu).$$

Substituting the corresponding expression N and taking into account the dependence of the Green's function on μ , we obtain:

$$F = \int_0^\mu d\mu 2iV \lim_{t \rightarrow -0} \int \frac{d\vec{p} d\omega}{(2\pi)^4} \exp(-i\omega t) G(\vec{p}, \omega).$$

A somewhat more complicated example is the current, which is expressed in terms of field operators [18, 19] as follows:

$$\vec{J}(\vec{r}) = \frac{ie}{2m} \sum_a \langle (\nabla \psi_a^*(\vec{r})) \psi_a(\vec{r}) - \psi_a^*(\vec{r}) (\nabla \psi_a(\vec{r})) \rangle.$$

The current can also be expressed in terms of Green's functions as follows:

$$\vec{J}(\vec{r}) = \frac{ie}{2m} \sum_a \lim_{t \rightarrow -0} \lim_{\vec{r}' \rightarrow \vec{r}} (\nabla_{\vec{r}'} - \nabla_{\vec{r}}) [\pm G_{aa}(\vec{r}, t; \vec{r}', 0)].$$

The formalism developed above is restricted to the properties of many-particle systems at $T = 0$ (i.e., in the ground state). This is due to the fact that the ground state is always non-degenerate, which allows one to apply the adiabatic hypothesis: if the interaction is switched on slowly and then switched off slowly, this results only in the appearance of a phase factor, which ultimately cancels out. From a physical point of view, however, one cannot restrict the analysis to the case $T = 0$. In principle, the averaging in the definition of the Green's function can be performed over any quantum state or ensemble of states, and its analytical properties can be established following the same procedure as for $T = 0$. For example, one can define equilibrium Green's functions at finite temperature.

The general definitions of causal, retarded, and advanced single-particle Green's functions [20]

$$G_{ab}(x_1, t_1; x_2, t_2) = -i \text{Tr} [\hat{\rho} T \psi_a(x_1, t_1) \psi_b^*(x_2, t_2)];$$

$$G_{ab}^R(x_1, t_1; x_2, t_2) = -i \text{Tr} [\hat{\rho} T \psi_a(x_1, t_1) \psi_b^*(x_2, t_2) \pm \psi_b^*(x_2, t_2) \psi_a(x_1, t_1)] K(t_1 - t_2);$$

$$G_{ab}^A(x_1, t_1; x_2, t_2) = -i \text{Tr} \left[\hat{\rho} T \psi_a(x_1, t_1) \psi_b^*(x_2, t_2) \pm \psi_b^*(x_2, t_2) \psi_a(x_1, t_1) \right] K(t_2 - t_1)$$

remain valid for equilibrium states at finite temperature, when the statistical operator has the standard form of the grand canonical Gibbs distribution:

$$\hat{\rho} = \exp \left\{ -\frac{1}{T} (H - \Omega) \right\} = \sum_s \exp \left\{ -\frac{1}{T} (E_s - \mu N_s - \Omega) \right\} |s\rangle \langle s| = \sum_s \rho_s |s\rangle \langle s|.$$

The grand partition function is given by:

$$Z = \exp \left(-\frac{\Omega}{T} \right) = \text{Tr} \left[\exp \left(-\frac{H}{T} \right) \right],$$

where Ω is the grand thermodynamic potential. In the isotropic homogeneous case,

$$G_{ab}(x_1, t_1; x_2, t_2) = \delta_{ab} G_{ab}(x_1 - x_2, t_1 - t_2).$$

Further, at zero temperature, it is necessary to include the matrix elements of the field operators between all states of the system, since all excited states now contribute with nonzero weight. As a result, we obtain the following expression for the causal Green's function:

$$G(\vec{p}, \omega) = \frac{1}{2} (2\pi)^3 \sum_m \sum_n \rho_n \sum_a |\langle n | \psi_a | m \rangle|^2 \delta(\vec{p} - (\vec{P}_m - \vec{P}_n)) \frac{1}{\omega - \omega_{nm}}, \quad (10)$$

$$\omega_{nm} = E_m - \mu N_m - (E_n - \mu N_n).$$

By separating the real and imaginary parts of Eq. (10) for real frequencies using the Sokhotski–Plemelj formula, we obtain:

$$\text{Re} G(\vec{p}, \omega) = \frac{1}{2} (2\pi)^3 P \sum_m \sum_n \rho_n \sum_a |\langle n | \psi_a | m \rangle|^2 \delta(\vec{p} - (\vec{P}_m - \vec{P}_n)) \left[1 \pm \exp \left(-\frac{\omega_{nm}}{T} \right) \right] \frac{1}{\omega - \omega_{nm}};$$

$$\text{Im} G(\vec{p}, \omega) = -\pi \frac{1}{2} (2\pi)^3 P \sum_m \sum_n \rho_n \sum_a |\langle n | \psi_a | m \rangle|^2 \delta(\vec{p} - (\vec{P}_m - \vec{P}_n)) \left[1 \pm \exp \left(-\frac{\omega_{nm}}{T} \right) \right] \delta(\omega - \omega_{nm}).$$

An important property of Eq. (10) is that, if an approximation for the self-energy function of finite order is substituted into it, the resulting approximation for the Green's function becomes equivalent to summing an infinite series of perturbation terms. This yields a significantly better result than a direct term-by-term calculation of the initial perturbation series. This is a natural consequence of the self-consistent approach. Another, less favourable consequence is that any approximate self-energy function must be verified for consistency with the general analytical properties of the Green's function (which follow from the principle of causality).

Conclusion

Research on macro systems using the Green's function method is a powerful mathematical tool that finds application in various fields of theoretical and applied physics. This method allows for the description of complex systems with numerous interacting particles or elements. In accordance with the subject of this work, Green's functions for macroscopic systems and ideal Fermi liquids, as well as two-particle Green's functions, finite-temperature Green's functions, and the Dirac equation, have been considered. The relevance of this study is associated with the application of Green's function methods to quantum phenomena. Indeed, this method is considered one of the most effective approaches for solving partial differential equations corresponding to various physical processes. In particular, when solving the governing equations of such phenomena, an auxiliary partial differential equation involving the Green's function in operator form is first solved. After this procedure, by passing to the momentum representation, Fourier transforms with respect to complex variables are performed. Then, within the framework of zeroth- and first-order approximations, integral expressions are obtained, and the limits of integration are chosen depending on the specific features of the process under consideration. Performing this integration makes it possible to obtain explicit solutions describing the properties of the system in which the given phenomenon occurs. Within this approach, methods of mathematical physics and special functions play a key role, which becomes particularly evident in large-scale calculations. The mathematical derivations presented in this paper have been given in a simplified form. The main emphasis has been placed on demonstrating the features of applying Green's function methods to the solution of equations describing quantum effects in macroscopic systems. The obtained results can be used in the following areas of theoretical and mathematical physics: studying the dynamics of fields, particle interactions (for example, in quantum electrodynamics and quantum chromodynamics), and scattering

processes; calculating the energy spectrum of electrons and phonons in crystals, studying superconductivity (Gorkov equations), magnetism, and the behavior of semiconductors; analyzing phase transitions, critical phenomena, and non-equilibrium processes; and calculating the propagation of waves in heterogeneous media, turbulence, and the response of hydrodynamic systems to external disturbances.

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Макроскоптық кванттық жүйелердің Грин функцияларының қасиеттері мен ерекшеліктерін зерттеу

Макроскоптық жүйелерде өтетін процестер Грин функциялары арқылы зерттелді. Идеалды ферми-сұйықтың, бір және екі бөлшектік, шекті температуралық, Дирак және Шредингер теңдеулерінің Грин функциялары шығарылып көрсетілді. Көп бөлшектік кванттық жүйелерге арналған теңдеулердің шешімдерін Грин функциясы тәсілімен анықтаудың жолдары қарастырылған. Операторлар мен шекаралық шарттарға арналған Грин функцияларының көмегімен сұйыққа арналған теңдеулердің шешімдері интегралдық түрлерде ұсынылды. Процестерге Грин функциясын дербес туындылы дифференциалдық теңдеулерді шешуге, Фурье тәсілін қолдануға, интегралдық түрлендірулер жасауға, арнайы функциялар мен математикалық талдау тәсілдерін пайдалануға тура келеді. Дифференциалдық теңдеуден эквивалентті интегралдық теңдеуге көшу жолдары келтіріліп, соңғы теңдеудің шешімі Грин функциясы бойынша анықталды. Бұл жағдай импульстік бейнеге ауысу арқылы және комплекстік айнымалылар бойынша жүргізілетін Фурье түрлендірулерін қолдану арқылы іске асырылды. Тұрақты толқындық функциялары бар макроскоптық жүйелер гейзенберттік бейнелеуде қарастырылды. Әрине бұл бейнелеуде физикалық шамаларға сәйкес қойылатын операторлар уақытқа айқын түрде тәуелді болмайтындығы белгілі. Барлық есептеулер сыртқы магнит өрісі ескерілген және ескерілмеген жағдайларға сәйкес орындалды.

Кілт сөздер: ферми-сұйық, Фурье түрлендіруі, импульстік бейнелеу, макроскоптық жүйе, Грин функциясы, интегралдық теңдеу

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Исследование особенностей и свойств функций Грина макроскопических квантовых систем

Процессы в макроскопических системах исследовались с помощью функций Грина. Показаны функции Грина для идеальной ферми-жидкости, одной и двух частиц, предельной температуры и уравнений Дирака, Шредингера. Рассмотрено решение уравнений, описывающих процессы в многочастичных квантовых системах, методом функций Грина. С помощью функции Грина для оператора и граничных условий решение уравнения для жидкости записано в интегральном виде. При применении функции Грина к процессам возникает необходимость решения дифференциальных уравнений частных производных, применения метода Фурье, интегральных преобразований, специальных функций и методов математического анализа. Использован переход от дифференциального уравнения к эквивалентному интегральному уравнению и показано решение данного уравнения с функцией Грина. При этом был осуществлен переход на импульсное представление и преобразования Фурье через комплексные переменные. Рассмотренные макроскопические системы с постоянными волновыми функциями были описаны на основе гейзенберговских представлений, где операторы, соответствующие физическим величинам, явно зависят от времени. Расчеты производились с учетом и без учета влияния внешних полей.

Ключевые слова: ферми-жидкость, преобразование Фурье, импульсное представление, макроскопическая система, функция Грина, интегральное уравнение

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