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Evaluation of Relativistic Plasma Dispersion Functions Using Downward Recursion and Analytical Relations

The Dnestrovskii functions provide a powerful framework for understanding wave propagation, attenuation and instabilities in plasmas where relativistic effects dominate, as relativistic plasma dispersion functions. To calculate the Dnestrovskii functions for a wide range of parameter values, effective analytical and downward recurrence formulae are proposed in this study. These formulae allow users to make efficient calculations specific to high-energy or relativistic plasmas. Because the recurrence formulas are simple, calculating time and accuracy are improved, and the formulas are easy to use. The results obtained using the new analytical and downward recurrence formulae agree well with published results and those obtained using numerical calculation methods for a wide range of parameters.

Keywords: Dnestrovskii functions, downward recurrence relations, relativistic plasma dispersion functions, plasma physics

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Introduction

Relativistic plasma dispersion functions (PDFs) are essential for the study of wave propagation and instabilities in high-energy plasmas. They generalize non-relativistic plasma theory to account for relativistic effects, which are critical in environments where particles approach the speed of light. These functions are highly applicable in astrophysics, fusion research and intense laser-plasma interactions [1–5]. The PDFs, often referred to as Dnestrovskii functions, are particularly applicable to high-energy or relativistic plasmas [3]. These functions are critical for analyzing the behavior of wave propagation, damping and instabilities in plasmas where particle velocities approach a significant fraction of the speed of light. The Dnestrovskii functions provide a way of describing the relativistic effects on the plasma response to electromagnetic fields and waves [1–20].

The Dnestrovskii functions are typically defined by integrals involving the relativistic particle distribution function and the wave-particle interaction terms. Although closed-form analytical expressions are rare, these functions are essential for solving wave dispersion relations in relativistic plasmas, growth rates for instabilities in both thermal and non-thermal plasmas, energy transfer between particles and waves (e.g., for Landau and cyclotron damping in relativistic plasmas) [7–10].

High energy environments such as pulsar magnetospheres, relativistic jets and accretion disks around black holes all involve relativistic plasmas where Dnestrovskii functions are used to describe the behavior of waves and instabilities. In some fusion devices, especially in advanced scenarios involving high-energy particle beams, relativistic effects become important and Dnestrovskii functions are used to analyze plasma stability and wave-particle interactions. These functions extend classical plasma theory into the relativistic regime, making them essential for the analysis of high-energy plasmas in both astrophysical and laboratory contexts [3, 4]. Numerous studies have been proposed for the evaluation of Dnestrovskii functions. These studies are widely used in applied sciences.

Dnestrovskii functions play a crucial role in describing weakly relativistic, magnetised and thermal plasmas [3, 4].These functions, together with the Shkarofsky functions, are particularly important for waves with small wave numbers that are perpendicular to the magnetic field [3]. Robinson [3, 5] has made an extensive study of these plasma dispersion functions, providing series expansions, asymptotic series, recurrence relations, integral forms and approximations.An approach to relativistic quantum plasmas, including limit cases, using covariant Wigner functions has been presented in [6]. Matroli has developed a numerical method for the calculation of the weakly relativistic Vlasov dielectric dyad in magnetised thermal plasmas [7]. Another study investigated processes and equilibria in relativistic thermal plasmas, including the effects of magnetic fields on thermal, Comptonised synchrotron emission and pair equilibria [8]. The equilibria of relativistic thermal plasmas have been studied, taking into account electron-positron creation and annihilation, and photons produced in the plasma [9]. Krivenski provided a simple and exact expression for the weakly relativistic plasma dispersion functions in terms of the Z-function to study the absorption and emission properties of a magnetized plasma [10]. The physical properties of a finite thermal plasma as a function of temperature, proton optical depth and proton density or radius were studied in [11]. A new representation of the dielectric tensor elements introduced in magnetized plasmas, including for the weakly relativistic case [12]. In addition, Robinson introduced a more general class of plasma dispersion functions for waves with arbitrary perpendicular wave numbers [3]. Xiao (1988) developed a formalism to evaluate the growth rate of electromagnetic R-mode waves in a relativistic magnetized plasma, showing that relativistic corrections reduce the wave growth rate compared to the non-relativistic approximation [13]. New results, presented by Robinson, on functions governing wave properties in non-relativistic and weakly relativistic plasmas [14]. Melrose extended this work by defining generalized Trubnikov functions for unmagnetized plasmas, deriving recursion relations to generate expressions in terms of the relativistic plasma dispersion function T(z, ρ) introduced by Godfrey et al. in 1975 [15]. Exact and moderately relativistic plasma dispersion functions for weakly relativistic, magnetized, thermal plasmas are presented in [16]. A closed set of Lorentz covariant fluid equations for relativistic magnetized plasmas, allowing for anisotropy and heat flow, derived by Hazeltine (2002). Analytical approaches to thermodynamic functions of dense, fully ionized, non-ideal electron-ion plasmas, including weakly relativistic and magnetized cases, are reviewed in [17]. Lyutikov discussed microphysical processes in magnetically dominated relativistic astrophysical plasmas, but does not mention Dnestrovskii functions [18]. The electrostatic dispersion relation in a magnetized plasma using a weakly relativistic quantum kinetic model based on the Dirac equation derived in [19]. A comprehensive review of quantum kinetic theory for modelling plasmas, including weakly relativistic, magnetized and thermal plasmas, given in [20]. The relaxation characteristics in dense plasmas have been studied from the effective potentials using the Coulomb logarithm in [21]. Mamedov not only derived upward and downward Dnestrovskii recurrence formulas, but also developed an effective formula for Dnestrovskii functions using the binomial expansion theorem, expressing them in terms of binomial coefficients and incomplete gamma functions [4]. While in most cases the relativistic plasma dispersion function does not have a simple closed-form expression, it can be computed numerically by integrating the relativistic particle distribution over velocity space, considering the appropriate relativistic dynamics. The evaluation of Dnestrovskii functions can be challenging due to their complexity and dependence on relativistic effects. These functions generally lack simple closed-form solutions, so a variety of analytical approximations, numerical integration and specialized computational methods are used to evaluate them.

In this study, the Dnestrovskii function is calculated using the downward recurrence relations and analytical formula. The new analytic formula for the Dnestrovskii function is obtained by the use of the exponential-integral function. The results obtained according to the downward recurrence relation and the analytical formula obtained are evaluated and compared with those found in the literature. It is seen that the proposed method is suitable in terms of time and accuracy of the calculations.

Recurrence and analytical relations for the Dnestrovskii functions

The Dnestrovskii functions for the integer and non-integer values of q and z are defined as follows [5],

$$F_q(z) = \frac{1}{\Gamma(q)} \int_0^\infty \frac{x^{q-1} e^x}{x+z} dx \tag{1}$$

$$F_{q+1/2}(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-x^2} F_q\left(z+x^2\right) dx \tag{2}$$

In the literature there is also a series representation of the Dnestrovskii functions in the following form [3],

$$F_q(z) = z^{q-1} e^z \Gamma(1-q) - \sum_{j=0}^\infty \frac{z^j \Gamma(1-q)}{\Gamma(j+2 \pm q)} \tag{3}$$

In case the value of z is real and positive, the following equation can be obtained [3],

$$F_q(z) = \sum_{j=0}^{\infty} \frac{L_j^{(1-g)} z}{j+1} \tag{4}$$

In this study an analytical formula for the Dnestrovskii function is given using the exponential integral equation, which is given as follows,

$$F_q(z) = e^z E_q(z) \Gamma(q) \tag{5}$$

where $E_q(z)$ is an exponential integral function which is defined as follows [22],

$$E_q(z) = \int_1^{\infty} \frac{e^{-zt}}{t^q} dt \tag{6}$$

Note that the Dnestrovskii functions satisfy the following downward recurrence relations for an integer order and for a non-integer value, respectively,

$$F_q(z) = \frac{1}{z}\left[1 - qF_{q+1}(z)\right] \tag{7}$$

$$F_{q+1/2}(z) = \frac{1}{z}\left[1 - \left(q + 1/2\right) F_{q+3/2}(z)\right] \tag{8}$$

The upward recurrence relations of the Dnestrovskii function for an integer and a non-integer value are obtained using Eq. (7) and Eq. (8) as follows,

$$F_{q+1}(z) = \frac{1}{q}\left(1 - zF_q(z)\right) \tag{9}$$

$$F_{q+3/2}(z) = \frac{1}{q + 1/2}\left(1 - zF_{q+1/2}(z)\right) \tag{10}$$

In the case that $q = 0$, the Eq. (1) and Eq. (2) take on the following form, respectively,

$$F_0(z) = \frac{1}{z} \tag{11}$$

$$F_{1/2}(z) = \sqrt{\frac{\pi}{z}} e^z Erfc\left(\sqrt{z}\right) \qquad \left|Arg(z)\right| < \pi \tag{12}$$

For the special case $q = 1$, Eq. (9) and Eq. (10) take the following forms

$$F_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt = e^z E_1(z) \qquad \left|Arg(z)\right| < \pi \tag{13}$$

$$F_{3/2}(z) = 2\left(1 - \sqrt{\pi z} e^z Erfc\left(\sqrt{z}\right)\right) \tag{14}$$

where $E_1(z)$ is the $q = 1$ case of the exponential integral function given in equation (6) and is defined as follows [21],

$$E_1(z) = \int_1^{\infty} \frac{e^{-zt}}{t} dt \tag{15}$$

and in Eq. (14) $Erfc(z)$ is referred to as the complementary error function, which is defined in [21] as,

$$Erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt \tag{16}$$

Results and discussion

In this study, the calculation of the Dnestrovskii function is carried out using the analytical formula and the downward recurrence relation. In addition, the results of the calculation with the formulas proposed in the literature by means of approximate methods are compared with the results obtained in the present study. New approximations have been proposed for the calculation of the Dnestrovskii functions that arise from the propagation of waves and from instabilities in high energy plasmas. Based on the downward recurrence relations and new formula obtained in this study (Eq. (5)), a program was developed to compute the Dnestrovskii functions using the Mathematica 10 programming language. Furthermore, the results of our calculations are compared with numerical and theoretical models from the literature. Given a sufficiently high value for q, it is possible to select any initial value. As mentioned above, all values for $F_{q+1/2}(z)$ and $F_q(z)$ functions have the same order, so we can use Eq. (10–13) as the starting point for the backwards calculation. It should also be noted that this study represents the first time in the literature that the results of calculating Eq. (2) and the downward recurrence relations are presented comparatively.

The precision of the calculation results and their accuracy are listed in Table 1, in accordance with (Eq. (5)), downward (Eq. (7)) and upward (Eq. (9)) recurrence formulas (for $F_q(z)$), the serial representation of the Dnestrovskii function (Eq. (3–4)), Mamedov's formula [4] and the Mathematica numerical integration method (Eq. (1)) for arbitrary integer and non-integer values of the q and z parameters. Table 2 shows the calculation results for arbitrary integer and non-integer values of the downward (Eq. (8)) and upward (Eq. (10)) recurrence formulas (for $F_{q+1/2}(z)$) and the numerical integration method of (Eq. (2)) for arbitrary integer and non-integer values of the parameters q and z. As can be seen from Table 1, the Eq. (5) gives results that are in good agreement and high precision with Eq. (1) for all positive and real values of q and z. On the other hand, according to the calculations, it is easy to see from the table that Eq. (7) does not give results for values of $q<z$, while Eq. (9) gives results compatible with the Eq. (1). Similarly, while Eq. (7) gives results compatible with the Eq. (1) for values of $q>z$, Eq. (9) does not give results for these values of q and z. Furthermore, this table shows that for values of $q<z$, Eq. (3) and Eq. (4) yield no results whatsoever. In the case where q is about ten times larger than z, all of the equations, except for Eq. (7), give results that agree with the Eq. (1). As can be seen in Table 1, Eq. (3) and Eq. (4), which are the serial representations of the Dnestrovskii function, provide computational results only for some limited values of q and z. The same trend is observed for the downward Eq. (7) and upward Eq. (9) recurrence relations of the Dnestrovskii function as well. The empty cells in Table 1 and Table 2 are left blank because the corresponding formulae fail to give correct results for the given values of q and z. It should also be noted that in the evaluation of the calculation results obtained in Table 2, the downward recurrence formula Eq. (7) demonstrates improved calculation accuracy when the upper limit (N) is chosen between 50 and 100. At these N values, the results obtained from the downward recurrence formula are closer to the values obtained from the Mathematica numerical integration results.

Table 2 compares the calculated results for non-integer q and z values of Eq. (8) and Eq. (10) recurrence formulae with the Mathematica numerical integration equation (Eq. (2)) calculations. According to the calculation results shown in Table 2, in cases where z is smaller than q, Eq. (8) gives no result. Eq. (8) gives a result when z is greater than eight times q, but does not give a result for $z>q$ values where this condition is not satisfied. In cases where the value of z is greater than 20 times the value of q, the downward recurrence formula Eq. (8) shows optimal performance when initiated at $N = 50$. Under these conditions, the results obtained from Eq. (8) are more consistent with the results obtained from the Mathematica numerical integral equation Eq. (2).The calculation results were evaluated for the values of q and z where both equations yield valid outputs. In these cases, the results obtained with both equations are in good agreement with the results obtained with the Mathematica numerical integral equation.

Table 1

The comparative values of the relativistic Dnestrovskii functions $F_q(z)$ for $N = 100$

q	z	Eq.(9)	Eq.(7)	Eq.(5)	Eq.(3)	Eq. (4)	Ref. [4] Eq. (14)	Mathematica numerical integration results (Eq.(1))
2,6	10,5	–	–	0.07738359185496	0.07738359319046	–	0.07740403292196	0.07738359177273
4,3	12,5	–	–	0.06036964974966	0.06036871671676	–	0.06034126851801	0.06036964975028
6,6	0,5	0.16144676029281	–	0.16144676029281	0.16144797096149	0.16144797282830	0.16144676029281	0.16144797096083
22,6	1,1	0.04395518198149	–	0.04395518198149	0.04395518198149	0.04395518199762	0.04395518198149	0.04395518199551
29,9	2,9	0.03135428198352	–	0.03135428198352	0.03135428198352	0.03135428263318	0.03135428198352	0.03135428198352
10,4	40,7	–	–	0.01964758195555	–	–	0.01964758185953	0.01964758195555
18,7	10,7	–	–	0.03474664148905	0.03475300967693	0.03474664141427	0.03444496554965	0.03474664148905
38,5	3,1	0.02446328774997	–	0.02458333183835	0.02458333183835	0.02458071572591	0.02458333183835	0.02458333183842
88,5	1,1	0.01128508276980	–	0.01128508276980	0.01128508276980	–	0.01128508276980	0.01128508276992
12,1	0,5	0.08586109345137	–	0.08586109345030	0.08586109345030	0.08586109345028	0.08586109345057	0.08586109345031
44,3	4,5	0.02087867730034	–	0.02087867730034	0.02087867730034	0.02083868126221	0.02087867730034	0.02087867730001
61,7	2,5	0.01581274164031	–	0.01581274164031	0.01581274164031	–	0.01581274164031	0.01581274164032
10,7	65,2	–	0.01319920773514	0.01319920773515	–	–	0.01319920773515	0.01319920773515
0,7	88,2	–	0.01124956878817	0.01124956878817	–	–	0.01124956878817	0.01124956878849
8,1	58,1	–	0.01513299426812	0.01513299427036	–	–	0.01513299427036	0.01513299427035
30,3	90	–	0.00832977806673	0.00832977806674	–	–	0.00832977806674	0.00832977806674
30,3	9	–	–	0.02594829066550	0.02594832874244	0.02594828422538	0.02594829040580	0.02594832874244
10,3	0,1	0.10624884726200	–	0.10624884726200	0.10624884726200	0.10624884726201	0.10624884726200	0.10624884726182
5,1	40,7	–	–	0.02188530948366	–	–	0.02188530948366	0.02188530948627
5,1	50,7	–	0.01794963288102	0.01794963143260	–	–	0.01794963143260	0.01794963143486
80,9	23,7	–	–	0.00963113191696	–	–	0.00963113191696	0.00963113193243
80,9	3,7	0.01195532365190	–	0.01195532365190	0.01195532365190	–	0.01195532365190	0.01195532367379
0,4	55,5	–	0.01789130056259	0.01789130056259	–	–	0.01789130056259	0.01789130056274
55,5	0,4	0.01821247505245	–	0.01821247505245	0.01821247505245	–	0.01821247505245	0.01821247505246
25,5	0,4	0.04013370619365	–	0.04013370619365	0.04013370619365	0.04013370632818	0.04013370619365	0.04013370619365
26,6	2,6	0.03534205349374	–	0.03534205349376	0.03534205349376	0.03534205314621	0.04013370619365	0.03534205349376
9,6	96,6	–	0.00942408325233	0.00942408325233	–	–	0.00942408325233	0.00942408325232
12,3	5,8	–	–	0.05732327768504	0.05732327768328	0.05732327800416	0.05672578677826	0.05732327768505
66,8	5,8	0.01395051136111	–	0.01395051136111	0.01395051136111	–	0.01395051136111	0.01395051136111
36,8	0,19	0.02778130570315	–	0.02778130570315	0.02778130570315	0.02778148200364	0.02778130570315	0.02778130570331
5,8	0,19	0.19853374482721		0.19856031044880	0.19856031044880	0.19856030610462	0.19856014492373	0.19856031046867
10.1	65.0	–	0.01323420640588	0.01323420640589	–	–	0.01323420640589	0.01323420640589
2.7	80	–	0.01209656583428	0.01209656583428	–	–	0.01209656583428	0.01209656580835
45	2	0.02171818885051	–	0.02171818885051	—	–	0.02171818885051	0.02171818885000
11	3	–	–	0.07552959181501	—	0.07552959189899	0.07550995055347	0.07552959178514
11.8	3	–	–	0.07129646176920	0.07129646176921	0.07129646178124	0.07130379335562	0.07129646176921
15	90	–	0.00953657711636	0.00953657711636	–	–	0.00953657711636	0.00953657711635
3.8	42	–	0.02225389874224	0.02225389874224	–	–	0.02225389874224	0.02225389874223
1.7	42	–	0.02290283961924	0.02290283961924	–	–	0.02290283961924	0.02290283961924

Table 2

The comparative values of the relativistic Dnestrovskii functions by using downward recurrence relations, $F_{q+1/2}(z)$ for $N = 100$

z	*q*	Eq. (10)	Eq. (8)	Mathematica numerical integration results (Eq.(2))
65,2	5,5	0.01415945096631	0.01415945956818	0.01415946020549
31,2	1,5	0.03062168076122	–	0.03062168043606
90,6	1,5	0.01074417463866	0.01074417463694	0.01074417401793
90,6	7,5	0.01019403295080	0.01020148444383	0.01020148477889
90,6	12,5	–	0.00971055000316	0.00971055061711
90,6	35,5	–	–	0.00794775825821
1,1	35,5	0.02806487203395	–	0.02806487252910
2,9	72,5	0.01343373410043	–	0.01343373434772
4,3	12,5	0.06219097218131	–	0.06219097408672
48,6	12,5	0.01643361558159	–	0.01642020642880
99,4	8,5	0.00944895864996	0.00927449889396	0.00927449887642
19,7	27,5	0.02144651625463	–	0.02144651245955
9,7	46,5	0.01805778324645	–	0.01805778305634
0,7	46,5	0.02163777350948	–	0.02163777340892
9,7	4,5	0.07188671251409	–	0.07188671200669
79,3	4,5	0.01194065842446	0.01194065842945	0.01194065838736
37,5	4,5	0.02386796432871	–	0.02386796430175
83,9	7,5	0.01095520717832	0.01095055813507	0.01095055849122
91,3	27,5	–	–	0.00843373526493
62,4	83,5	–	–	0.00688084291229
62,4	3,5	0.01518640814421	0.01518640814399	0.01518640803752
47,1	13,5	–	–	0.01656101572075
47,1	6,5	0.01869768084745	0.01865405371751	0.01869769133608
2,7	10,5	0.08003592842445	–	0.08044447582587
2,7	38,5	0.02483322536828	–	0.02483322945412
75,1	3,5	0.01272968856100	0.01272968865274	0.01272968861057
57	2,5	0.01659481082540	0.01659481082595	0.01659480990599

Figure 1 shows the graph of the data plotted against each other from the calculated results obtained using the Eq. (5) and the Mathematica numerical integration method for arbitrary q and z values. As can be seen from the graph, the results of the calculations obtained from the two methods are in good correlation with each other.

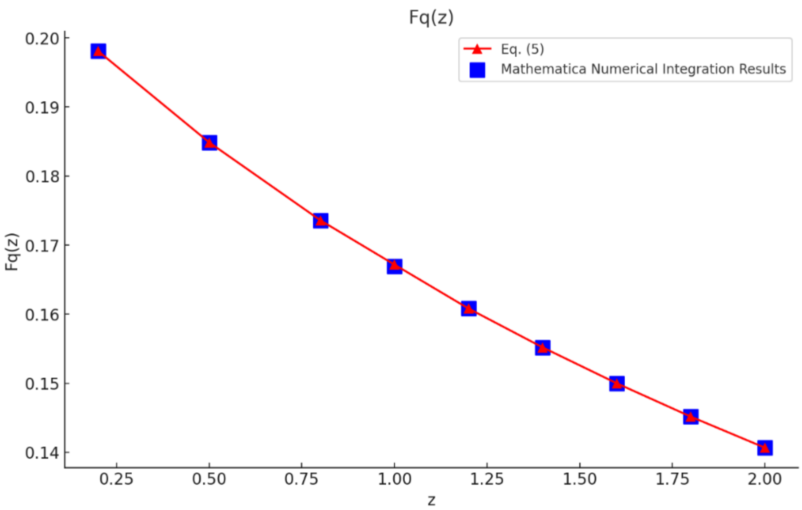

Figure 1. Correlation between Eq. (5) and Mathematica numerical integration results

Figure 2 shows 2D scatter plot that the calculation results of Eq. (5) and results of Mathematica numerical integration method are in a very good agreement. The ratio of the two sets of data to each other is centered around 1, as can be seen in the Figure 1.

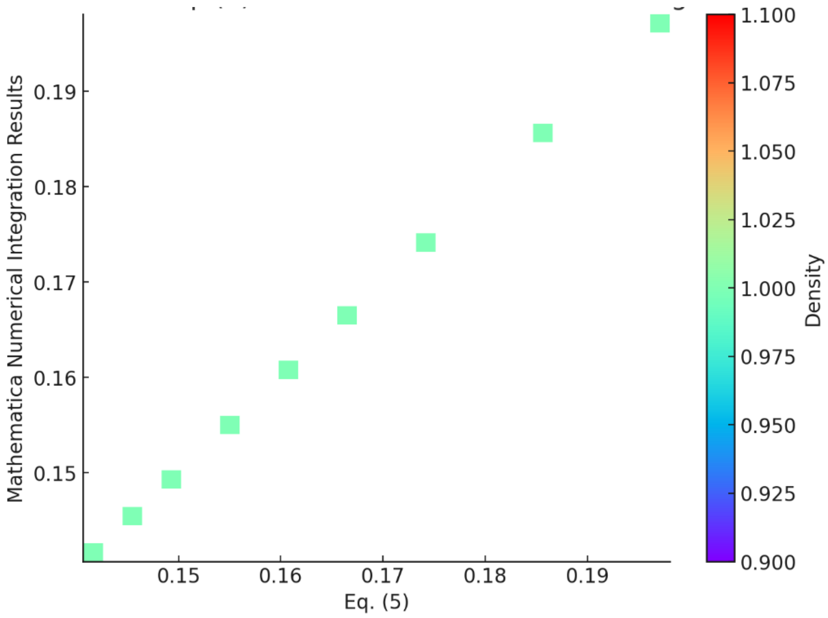

Figure 2. Comparison of the results obtained using the Eq. (5) and the Mathematica numerical integration method as a function of the parameter z for $q = 5.8$ and $N = 100$

Conclusion

The results obtained using the new formula (Eq. (5)), proposed in this study and downward recurrence formula (Eq. (7 & 8)), which allows the Dnestrovskii function to be calculated with high precision, agree very well with those obtained using the Mathematica numerical integration method. The accuracy of the proposed approaches for evaluating Dnestrovskii functions has been verified through comparison with other existing methods, demonstrating high consistency. These formulas, Eq. (5) and Eq. (7&8), provide a powerful computational framework for understanding wave propagation, damping and instabilities in plasmas where relativistic effects dominate. The performance of the Eq. (5) is better than previous ones and the formulas are easy to use.

The proposed analytical and recurrence formulas are particularly beneficial for specific plasma physics problems including: (i) wave-particle interaction calculations in fusion plasma heating scenarios, such as electron cyclotron resonance heating and radio-frequency wave absorption in tokamaks; (ii) dispersion relation analysis in astrophysical relativistic plasmas, including pulsar magnetospheres, relativistic jets, and accretion disk environments; and (iii) stability analysis and wave propagation studies in high-intensity laser-plasma interactions where weakly relativistic effects become significant. These methods provide computationally efficient alternatives to direct numerical integration, enabling faster parameter space exploration in plasma simulations.

Several constraints must be considered when applying these formulas. The downward recurrence relations (Eq. 7–8) exhibit limited parameter range validity. For integer orders, reliable results require $q>z$, while for half-integer orders, z must be approximately 8–20 times larger than q. Outside these ranges, the recurrence relations fail and numerical integration becomes necessary. The analytical formula (Eq. 5) demonstrates broader applicability but still requires careful initialization of the recurrence starting point ($N = 50$–100) to ensure numerical stability. Additionally, these functions are specifically designed for weakly relativistic, magnetized thermal plasmas with Maxwellian distributions, and therefore are not applicable to ultra-relativistic regimes, non-thermal particle distributions, or highly non-equilibrium plasma conditions.

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Релятивистік плазманың дисперсиялық функцияларын төмен қайтарым рекурсиясы мен аналитикалық қатынастар арқылы есептеу

Днестровский функциялары релятивистік әсерлер басым болатын плазмалардағы толқындардың таралуын, әлсіреуін және тұрақсыздықтарын сипаттау үшін тиімді теориялық негіз және релятивистік плазманың дисперсиялық функциялары ретінде қолданылады. Бұл зерттеуде параметрлердің кең ауқымы үшін Днестровский функцияларын есептеуге арналған тиімді аналитикалық өрнектер мен төмен қайтарым рекурренттік формулалар ұсынылады. Ұсынылған өрнектер жоғары энергиялы және релятивистік плазмаларға тән есептеулерді жоғары тиімділікпен орындауға мүмкіндік береді. Рекуррентік формулалардың қарапайымдылығы арқасында есептеу уақыты мен дәлдігі жақсарады және формулаларды қолдану жеңілдейді. Жаңа аналитикалық және төмен қайтарым рекуррентік формулалар арқылы алынған нәтижелер параметрлердің кең аймағы үшін бұрын жарияланған нәтижелермен және сандық есептеу әдістерімен алынған нәтижелермен жақсы сәйкестік көрсетеді.

Кілт сөздер: Днестровский функциялары, төмен қайтарым рекурренттік қатынастар, релятивистік плазманың дисперсиялық функциялары, плазма физикасы

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Оценка релятивистских дисперсионных функций плазмы с использованием рекуррентных формул нисходящего типа и аналитических соотношений

Функции Днестровского представляют собой эффективный теоретический аппарат для описания распространения волн, затухания и развития неустойчивостей в плазмах, где доминируют релятивистские эффекты, и выступают в качестве релятивистских дисперсионных функций плазмы. В настоящей работе предложены эффективные аналитические выражения и рекуррентные формулы нисходящего типа для вычисления функций Днестровского в широком диапазоне параметров. Данные выражения обеспечивают возможность высокоэффективных вычислений, характерных для высокоэнергетической и релятивистской плазмы. Благодаря простоте рекуррентных формул достигаются улучшенное вычислительное время и точность по сравнению с предыдущими методами, а также обеспечивается простота применения. Полученные результаты на основе предложенных аналитических выражений и рекуррентных формул нисходящего типа находятся в хорошем согласии как с опубликованными данными, так и с результатами, полученными с помощью численных методов, в широком диапазоне параметров.

Ключевые слова: функции Днестровского, рекуррентные соотношения нисходящего типа, релятивистские дисперсионные функции плазмы, физика плазмы

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